Chorale Synthesis by the Multidimensional Scaling of Pitches

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ABSTRACT

This paper outlines a unique algorithmic model of chorale synthesis based on mathematics, speciﬁcally on algebra and geometry. There were four distinct stages in its development, 1. algebraic formulæ (1978) for the quantiﬁcation of the harmonicity of pitch intervals and of the rhythmic relevance or indispensability of the pulses in a multiplicative meter, 2. the computer program package Autobusk (1986) using the harmonicity formula for the interpretation of a pitch set or musical scale as a ratio matrix, 3. multidimensionally scaling (2001) a scale ratio matrix into a Cartesian chart of two or more dimensions and 4. chorale synthesis (2012) by rules based on the multidimensional scaling of pitch sets and on the pulse indispensability formula. These four stages will each be outlined in a separate section.

1. INTRODUCTION

In 1975, while preparing to compose a large microtonal piano piece (see section 2), I was faced with the problem of how to treat quarter-tones harmonically. Hindemith [1] had referred to the harmonic relevance of twelve-tone chromatic intervals but not to intervals outside this set. Partch [2] listed and commented on several just-intoned intervals but not in the context of functional harmony. But both authors explicitly based their investigations on interval ratios and on prime numbers contained therein. It soon became clear to me that I would have to do my own investigations along these lines.

2. ALGEBRAIC FORMULÆ

This section concerns theoretical work done in 1975-78 in connection with the writing of *Çoğluotobüsişle*tmesi, a thirty-minute work for microtonally retuned piano. My aim was to determine – in the micro- and macrotemporal (frequency and time) domains, respectively – the extent to which an element (pitch or pulse) of a scale or meter contributes to setting up a tonal or metric ﬁeld of speciﬁc strength. The ﬁeld-strength would be controlled by the gradient of a straight line in a two-dimensional space in which the x-axis represents the said contribution in terms of interval harmonicity or pulse indispensability and the y-axis the probability of the element in a stochastic compositional environment. A horizontal line would yield atonality/ametricism with all elements equally probable; increasing the gradient would raise the ﬁeld-strength – see Figure 1.

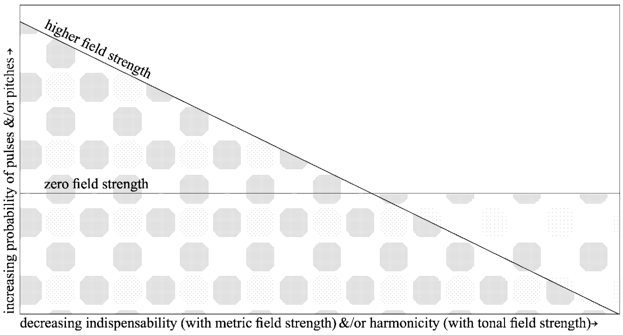

Figure 1. A straight line, its gradient determining tonal or metric ﬁeld strength by relating element probabilities to interval harmonicity or pulse indispensability.

2.1 Numerical Indigestibility, Interval Harmonicity

For this piece, the notes D, E, F-sharp and B are retuned in every octave of the piano a quarter-tone downwards. In this tuning, a total of 84 heptatonic scales containing three interval sizes – whole steps (3, 4 or 5), half steps (0, 1 or 2) and three-quarter steps (0, 2 or 4) – were found; other interval sizes were not used. In order to study the harmonic properties of these scales, it was necessary to ﬁrst study their component intervals. I turned to the old adage attributed to Pythagoras, that the smaller the numbers forming interval ratios (e.g. 1:2, 2:3, 3:4, 3:5 etc.), the more harmonic these are. 2½ millenia later, Partch was practically stating the same thing. However, it did not make sense to me that intervals like 6:7 and 7:8, basically unused in pre-20th Century Western (or even Indian) Music, be more harmonic than 8:9 or 9:10, two well-known whole tones. I directed my attention to the primes contained in the ratio numbers and came up, inspired by Euler's φ or function, with a formula for what I termed numerical indigestibility: the larger, "less digestible" primes inhibit the intervals' harmonicity, the formula for which I based on that for indigestibility. Harmonicity, a psychological phenomenon, has been long confused with physiological sensory consonance, itself deriving from the consistency of the basilar membrane as described in a legendary text by Plomp and Levelt [3]. The formulæ for indigestibility and harmonicity can be seen on the next page in Figure 2, with tangible results in Table 1. These items were ﬁrst published in [4] and most recently in [5].

$$\xi(N) = 2\sum_{r=1}^{\infty}\left(\frac{n_r(p_r-1)^2}{p_r}\right)$$

whereby:

1. $N = \prod_{r=1}^{\infty} p_r^{\,n_r}$
2. $N,\ n,\ p \in$ natural numbers
3. $p \in$ prime numbers

$$H(P,Q) = \frac{\operatorname{sgn}\big(\xi(Q)-\xi(P)\big)}{\xi(P)+\xi(Q)}$$

whereby $\operatorname{sgn}(x)=-1$ for $x<0$, else $\operatorname{sgn}(x)=+1$

Figure 2. Formulæ for the indigestibility ξ of whole number N (left) and the harmonicity H of interval ratio P:Q (right, Q>P).

N	ξ (N)
1	0.000000
2	1.000000
3	2.666667
4	2.000000
5	6.400000
6	3.666667
7	10.285714
8	3.000000
9	5.333333
10	7.400000
11	18.181818
12	4.666667
13	22.153846
14	11.285714
15	9.066667
16	4.000000

Interval-size (Ct)	Prime Decomposition 2	3	5	7	Number-ratio	Harmonicity
0.000	0	0	0	0	1:1	+∞
70.672	−3	−1	+2	0	24:25	+0.054152
111.731	+4	−1	−1	0	15:16	−0.076531
182.404	+1	−2	+1	0	9:10	+0.078534
203.910	−3	+2	0	0	8:9	+0.120000
231.174	+3	0	0	−1	7:8	−0.075269
266.871	−1	−1	0	+1	6:7	+0.071672
294.135	+5	−3	0	0	27:32	−0.076923
315.641	+1	+1	−1	0	5:6	−0.099338
386.314	−2	0	+1	0	4:5	+0.119048
407.820	−6	+4	0	0	64:81	+0.060000
427.373	+5	0	−2	0	25:32	−0.056180
435.084	0	+2	0	−1	7:9	−0.064024
470.781	−4	+1	0	+1	16:21	+0.058989
498.045	+2	−1	0	0	3:4	−0.214286
519.551	−2	+3	−1	0	20:27	−0.060976
568.717	−1	−2	+2	0	18:25	+0.052265
582.512	0	0	−1	+1	5:7	+0.059932
590.224	−5	+2	+1	0	32:45	+0.059761
609.776	+6	−2	−1	0	45:64	−0.056391
617.488	+1	0	+1	−1	7:10	−0.056543
680.449	+3	−3	+1	0	27:40	+0.057471
701.955	−1	+1	0	0	2:3	+0.272727
729.219	+5	−1	0	−1	21:32	−0.055703
764.916	+1	−2	0	+1	9:14	+0.060172
772.627	−4	0	+2	0	16:25	+0.059524
792.180	+7	−4	0	0	81:128	−0.056604
813.686	+3	0	−1	0	5:8	−0.106383
884.359	0	−1	+1	0	3:5	+0.110294
905.865	−4	+3	0	0	16:27	+0.083333
933.129	+2	+1	0	−1	7:12	−0.066879
968.826	−2	0	0	+1	4:7	+0.081395
996.090	+4	−2	0	0	9:16	−0.107143
1017.596	0	+2	−1	0	5:9	−0.085227
1088.269	−3	+1	+1	0	8:15	+0.082873
1129.328	+4	+1	−2	0	25:48	−0.051370
1137.039	−1	+3	0	−1	14:27	−0.051852
1200.000	+1	0	0	0	1:2	+1.000000

Table 1. The indigestibility of the natural numbers 1-16 (boxed, left) and all intraoctavic intervals upwards of absolute harmonicity threshold 0.05.

The 84 scales of *Çoğluotobüsişle*tmesi were now tuned to intervals taken within a xed tolerance from a table such as the one above while lowering the harmonicity threshold to include the prime number 11. The tuning was used to evaluate harmonicities which engendered probability values as shown in Figure 1. Table 2 shows an excerpt of the tuning as published in [4].

SCALE	I	II	III	IV	V	VI	VII
[C D-E-F+G A A#]	1/1	35/32	100/81	25/18	3/2	5/3	16/9
[D-E-F+G A A#C]	1/1	9/8	5/4	11/8	14/9	18/11	11/6
[E-F+G A A#C D-]	1/1	9/8	11/9	11/8	16/11	18/11	9/5
[F+G A A#C D-E-]	1/1	27/25	11/9	9/7	36/25	128/81	16/9
[G A A#C D-E-F+]	1/1	9/8	6/5	4/3	36/25	81/50	11/6
[A A#C D-E-F+G]	1/1	16/15	32/27	9/7	36/25	81/50	16/9
[A#C D-E-F+G A]	1/1	9/8	100/81	25/18	25/16	27/16	15/8
[C D-E-F+G A B-]	1/1	27/25	11/9	11/8	3/2	27/16	11/6
[D-E-F+G A B-C]	1/1	9/8	81/64	25/18	25/16	27/16	50/27
[E-F+G A B-C D-]	1/1	9/8	100/81	25/18	3/2	18/11	9/5
[F+G A B-C D-E-]	1/1	27/25	11/9	4/3	36/25	128/81	16/9
[G A B-C D-E-F+]	1/1	9/8	11/9	4/3	36/25	81/50	11/6
[A B-C D-E-F+G]	1/1	27/25	32/27	9/7	36/25	81/50	16/9
[B-C D-E-F+G A]	1/1	27/25	32/27	4/3	3/2	81/50	11/6

Table 2. Tuning of 14 of 84 scales as used in the piano piece *Çoğluotobüsişle*tmesi (1978) – the rst seven and the last seven each belong to a common cyclic mode.

2.2 Pulse Indispensability

Rhythms in *Çoğluotobüsişletmesi* were also computed by the method outlined in Figure 1. A total of six meters in 14 tempi ranging from MM 60 to 135 were taken for the piece, each meter denoted by multiplicative strati cation, e.g. “2x5” meaning 2 beats of 5 pulses each. A system, algebrized as a formula [6] and computer-programmed, (see below) allots each pulse of each meter a unique indispensability value ranging from zero to one less than the number of pulses, e.g. for 2x5: [9 0 6 3 4 8 1 7 3 5] – the bigger the number, the more indispensable the pulse. The indispensabilities, converted into probabilities, made rhythms with random numbers. Figure 3 shows the three-layered meters 3x2x2, 2x3x2 and 2x2x3 (better known as $^{3}/_{4}$, $^{6}/_{8}$ and $^{12}/_{16}$) with pulse indispensabilities.

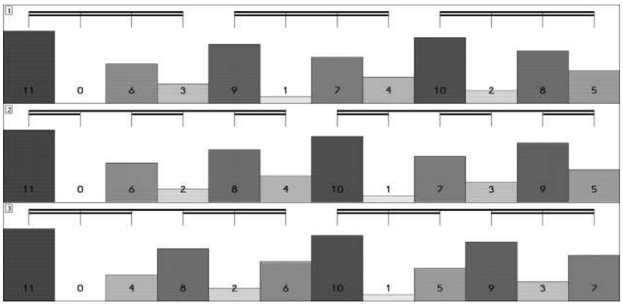

Figure 3. Pulse indispensabilites simultaneously shown as numbers, bar charts and shading for three meters on the third level of strati cation (with 12 pulses each).

3. AUTOBUSK

From 1986, the above-mentioned formulæ and related algorithms were programmed into the software package Autobusk running on an Atari ST computer, in which twelve parameters are applied to scales de ned in cents and meters as strati cations. Figure 4 shows a screen shot of the main program. The package together with a tutorial [7] can be freely downloaded from Mainz University:

<http://www.musikwissenschaft.unimainz.de/Autobusk/>

The rst piece composed with Autobusk was variazioni e un pianoforte meccanico (1986) for pianist and player piano. In 2000 Autobusk was declared completed.

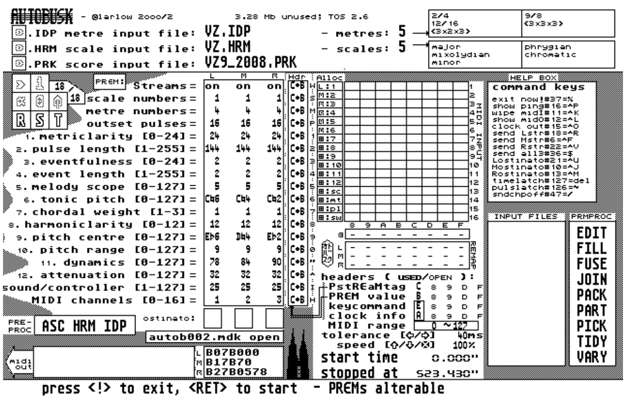

Figure 4. Screen shot of Autobusk main program.

One of Autobusk's peripheral programs, "HRM", derives a two-dimensional matrix of ratios from a musical scale de□ned in cents, e.g. the one-octave chromatic scale in Figure 5, with three constraints: minimum harmonicity, nominal tolerance (the width of a Gaussian bell placed on each scale degree to damp outlying ratios' harmonicities) and tuning alternatives (the number of ratios competing to represent each scale degree, e.g. 5/4 and 81/64 for the major third, 400 cents). The overall harmonicity for every possible tuning given the number of alternatives is evaluated (e.g. 28 tunings for 2 alternatives in an 8-note major scale) and the one with the highest value is selected.

Scale degree:	1	2	3	4	5	6	7	8	9	10	11	12	13
Input size (Ct):	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Tuning:	1/1	16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	9/5	15/8	2/1
Deviation (Ct):	0	+12	+4	+16	−14	−2	−10	+2	+14	−16	+18	−12	0
Note name:	C	D♭	D	E♭	E	F	F♯	G	A♭	A	B♭	B	C

12 tone matrix:

2→	3→	4→	5→	6→	7→	8→	9→	10→	11→	12→	13→	
16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	9/5	15/8	2/1	→1
−0.077	+0.120	−0.099	+0.119	−0.214	+0.060	+0.273	−0.106	+0.110	−0.085	+0.083	+1.000	
	135/128	9/8	75/64	5/4	675/512	45/32	3/2	25/16	27/16	225/128	15/8	→2
	+0.047	+0.120	+0.047	+0.119	+0.034	+0.060	+0.273	+0.060	+0.083	+0.040	+0.083	
		16/15	10/9	32/27	5/4	4/3	64/45	40/27	8/5	5/3	16/9	→3
		−0.077	+0.079	−0.077	+0.119	−0.214	−0.056	+0.057	−0.106	+0.110	−0.107	
			25/24	10/9	75/64	5/4	4/3	25/18	3/2	25/16	5/3	→4
			+0.054	+0.079	+0.047	+0.119	−0.214	+0.052	+0.273	+0.060	+0.110	
				16/15	9/8	6/5	32/25	4/3	36/25	3/2	8/5	→5
				−0.077	+0.120	−0.099	−0.056	−0.214	−0.050	+0.273	−0.106	
					135/128	9/8	6/5	5/4	27/20	45/32	3/2	→6
					+0.047	+0.120	−0.099	+0.119	−0.061	+0.060	+0.273	
						16/15	256/225	32/27	32/25	4/3	64/45	→7
						−0.077	−0.038	−0.077	−0.056	−0.214	−0.056	
							16/15	10/9	6/5	5/4	4/3	→8
							−0.077	+0.079	−0.099	+0.119	−0.214	
								25/24	9/8	75/64	5/4	→9
								+0.054	+0.120	+0.047	+0.119	
									27/25	9/8	6/5	→10
									−0.048	+0.120	−0.099	
										25/24	10/9	→11
										+0.054	+0.079	
											16/15	→12
											−0.077	

Figure 5. Ratio-harmonicity matrix of a chromatic scale octave as interpreted by program Autobusk/HRM.

As in *Çoğluotobüsişletmesi*, the harmonicity values are then converted into probabilities for the random selection of pitches.

4. MULTIDIMENSIONAL SCALING

Introduced about a half-century ago, multidimensional scaling is, to quote Wikipedia, "a means of visualizing the level of similarity of individual cases of a dataset." Writing "proximity" for "similarity", this means that if we know for instance the geographical distances between selected cities, it would be possible to construct a map in dimensions two or more (e.g. a globe) with the cities in the right place related to each other, except for the map being possibly rotated by a certain unpredictable angle or even laterally reversed.

In 2001, regarding harmonicity as a measure of the harmonic proximity of notes forming scales, I began to construct "maps" of the scales. Figure 6 shows a mapping of the chromatic scale matrix in Figure 5. Note the close proximity of the keynote (1:1) and the octave (1:2), and the mutually remote minor 2nd (15:16) and augmented 4th (32:45) or major 6th (3:5) and minor 7th (5:9).

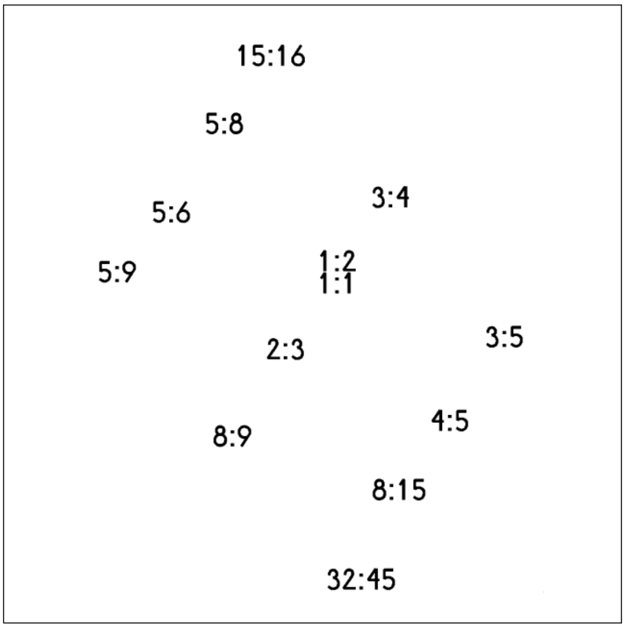

Figure 6. A multidimensionally scaled map of a chromatic scale octave as interpreted by Autobusk/HRM.

5. CHORALE SYNTHESIS

To quote Wikipedia again, a chorale "is a melody to which a hymn is sung by a congregation in a German Protestant Church service. The typical four-part setting of a chorale, in which the sopranos (and the congregation) sing the melody along with three lower voices, is known as a chorale harmonization. In certain modern usage, this term may include classical settings of such hymns and works of a similar character." Probably the most famous chorales in music history were written by J. S. Bach. While making visual pitch maps by multidimensional scaling, I wondered what the Bach chorales would look like if viewed in such maps. Accordingly, in 2012, I □nally got down to examining the chorale Zeucht ein zu deinen Toren ("Oh enter, Lord, thy temple") in this light. For a multidimensional scaling the piece would have to be tuned to just-intoned ratios (which I did manually), and the harmonicities determined by Autobusk. Figure 7 shows the chorale opening together with the ratios found. Note the four pairs of identical looking but differently tuned notes G, B in the bass staff and B, C in the treble.

Figure 7. A manually tuned rendition of the J. S. Bach chorale Zeucht ein zu deinen Toren (opening bars).

Figure 8 shows the multidimensionally scaled map of the Bach chorale opening bars' pitch material as manually tuned. The first chord is outlined in blue. The four pairs of identical looking differently tuned notes are numbered with lower case Roman numerals and are mutually quite remote.

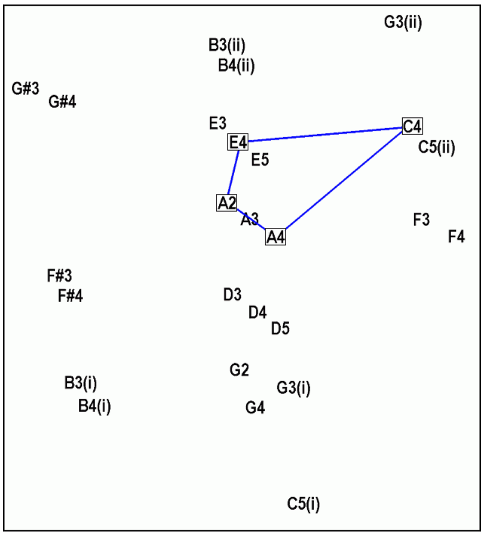

Figure 8. Multidimensionally scaled manually tuned pitches of the J. S. Bach chorale Zeucht ein zu deinen Toren (opening bars).

Every chord in the chorale was delineated as in Figure 8, and the images, one per chord, converted into a music-synchronized video. After intently observing the video, I came up with two simple rules for synthesizing a chorale:

1. The overall harmonicity of a chord randomly chosen from a multidimensionally scaled map is proportional to the indispensability of the pulse it occupies.
2. Every chord and the one succeeding share a note in common.

With these rules in mind, I began to work on a commissioned piece for computer-driven pipe organ entitled *Für Si*mon Jonassohn-Stein (the organ in question was housed in Cologne in the church of St. Peter, whose original name was Simon son of Jonas). The pitch material consisted of 79 just-intoned intervals spread over the full 4½-octave range of the organ, their ratios being prime-limit 7 (i.e. containing factors up to $2^{\pm6}$, $3^{\pm3}$, $5^{\pm1}$ and $7^{\pm1}$), and the minimum harmonicity was set at 0.07. Even though the organ's 54 half-steps were tuned to the regular 12-tone chromatic scale, the composition of the work was effected as though the pitches were just-intoned; this corresponds to the general practice of composing 12-tone tempered music with the harmony (but not the sound) of just intonation in mind.
Figure 9 shows a multidimensionally scaled map of the 79 pitches expressed as ratios, offering a total of 79079 different triads for random selection. Note the several individual four-note octave-chains.

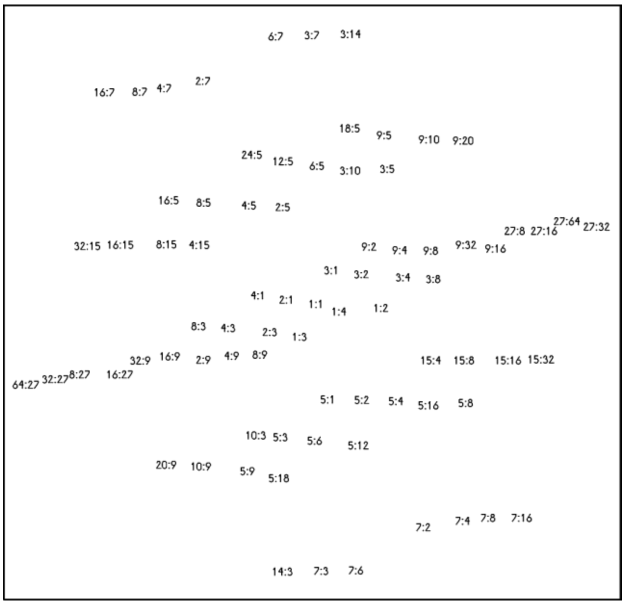

Figure 9. 79 multidimensionally scaled pitches selected for the organ piece *Für Simon Jonassohn*-Stein.

The meter was chosen for the piece was a slow 3x2, the half-note pulse indispensabilities being [5 0 3 1 4 2]. Using the method outlined above, four chorales were composed, partly interspersed and partly synchronized with Autobusk "improvisations" in the same harmonies as the chorales and in meters 2x2x2, 3x2x2, 2x3x2, 2x2x3, 2x2x2x2 and 2x2x2x3, whereby the fastest pulse is a 16th-note. Even though the work was performed in equal temperament, Figure 10 shows Chorale 1 with just-intoned ratios and cent deviations with which F*ür Simon* Jonassohn-Stein was composed.

Figure 10. Chorale 1 of *Für Simon Jona*ssohn-Stein.

6. SUMMARY

Unlike work done by many esteemed colleagues in the parallel field of musicology, *Für Simon Jona*ssohn-Stein does not attempt to re-create an existing style. Its idiom indeed sounds vaguely familiar, because of the tonal and metric elements employed. At the same time it also sounds somewhat singular, because of the unexpectedness of many of the chord changes. All functions and techniques described, from indigestibility, harmonicity, and indispensability to chorale synthesis, were developed as purely compositional tools, from *Çoğluotobüsişletmesi* and variazioni to the present. It is my hope that this paper – in addition to describing a continuous line of thought – also conveys my fascination by the extent to which solely mathematical, non-empirical means can synthesize an esthetically relevant music.

7. REFERENCES

[1] P. Hindemith, "The Craft of Musical Composition", Associated Music Publishers, 1942.

[2] H. Partch, "Genesis of a Music", University of Wisconsin, 1947.

[3] R. Plomp and W. J. M. Levelt, "Tonal Consonance and Critical Bandwidth", Journal of the Acoustical Society of America, 1965.

[4] C. Barlow, "Bus Journey to Parametron", Feedback Studio Cologne, 1980.

[5] C. Barlow, "On Musiquantics", University of Mainz, 2012.

[6] C. Barlow, "Two Essays on Theory", Computer Music Journal Vol. 11 No. 1, MIT Press, 1987.

[7] C. Barlow, "Autobusk", University of Mainz, 2000.